99.C (Abdilkadir Altintaş)

In the triangle ABC, the medians from A, B, C meet the sides BC, AC, AB at A_1, B_1, C_1. Also, the internal angle bisectors of angles A, B, C meet the sides BC, AC, AB at A_2, B_2, C_2. Show that the area of triangle $A_2B_2C_2$ is never greater than the area of triangle $A_1B_1C_1$.

Almost all solvers of this popular problem on geometrical inequalities argued as follows. If $\Delta = [ABC]$ denotes the area of triangle ABC, then $[A_1B_1C_1] = \frac{1}{4}\Delta$. In Figure 1, the angle bisector theorem shows that $AB_2 = \dfrac{bc}{a + c}$ and $AC_2 = \dfrac{bc}{a + b}$ so that $\Delta_A = [AB_2C_2]$ is given by $\Delta_A = \dfrac{1}{2}\dfrac{b^2c^2 \sin A}{(a + c)(a + b)} = \dfrac{bc}{(a + c)(a + b)}\Delta$ with similar expressions for Δ_B and Δ_C.

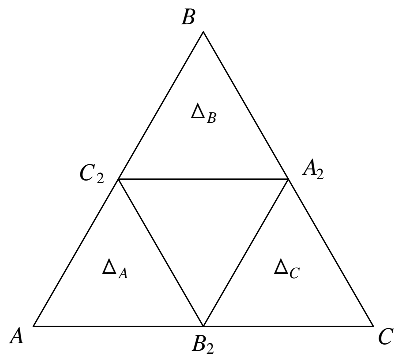

FIGURE 1

Then $[A_2B_2C_2] = \Delta - \Delta_A - \Delta_B - \Delta_C = \dfrac{2abc}{(a+b)(a+c)(b+c)}\Delta$ on substituting for Δ_A, Δ_B, Δ_C and simplifying.

But $\dfrac{2abc}{(a + b)(a + c)(b + c)} \leqslant \dfrac{1}{4}$, either by using the AM-GM inequality, $a + b \geqslant 2\sqrt{ab}$, etc. or by algebraic rearrangement:

$$\frac{1}{4} - \frac{2abc}{(a+b)(a+c)(b+c)} = \frac{a(b-c)^2 + b(a-c)^2 + c(a-b)^2}{4(a+b)(a+c)(b+c)} \geqslant 0.$$

Thus $[A_2B_2C_2] \leqslant \frac{1}{4}\Delta = [A_1B_1C_1]$ with equality if, and only if, $a = b = c$.

Triangle $A_2B_2C_2$ corresponds to the incentre of triangle ABC and the following proved the same result as **99.C** for the orthocentre (Martin Lukarevski) and the Gergonne point (R. F. Tindall). But the definitive generalisation was given by the GCHQ Problem Solving Group and Peter Nüesch. Consider the triangle $A'B'C'$ formed by the arbitrary concurrent Cevians shown in Figure 2 with $B'A : B'C = \lambda : 1 - \lambda$, $A'C : A'B = \mu : 1 - \mu$ and $C'B : C'A = \nu : 1 - \nu$.

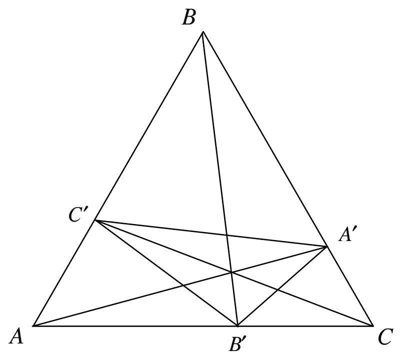

FIGURE 2

Then

$$[A'B'C'] = [1 - \mu(1-\lambda) - \nu(1-\mu) - \lambda(1-\nu)]\,\Delta$$
$$= (1 - \Sigma\lambda + \Sigma\lambda\mu)\,\Delta.$$

But, by Ceva's theorem, $\lambda\mu\nu = (1-\lambda)(1-\mu)(1-\nu)$ or $1 - \Sigma\lambda + \Sigma\lambda\mu = 2\lambda\mu\nu$.

So

$$[A'B'C'] = 2\lambda\mu\nu\Delta = 2\sqrt{\lambda(1-\lambda)\mu(1-\mu)\nu(1-\nu)}\Delta \leqslant \tfrac{1}{4}\Delta,$$

since $\lambda(1-\lambda) \leqslant \frac{1}{4}$, etc. There is equality if, and only if, $\lambda = \mu = \nu = \frac{1}{2}$ when $A'B'C'$ coincides with $A_1B_1C_1$.

Correct solutions were received from: R. G. Bardelang, M. Bataille, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, A. P. Harrison, G. Howlett, M. Lukarevski, J. A. Mundie, Peter Nüesch, G. Strickland, K. B. Subramaniam, E. Swylan, A. Tee, I. Timmins, R. F. Tindall, G. B. Trustrum and the proposer A. Altintaş.

99.D (John D. Mahony)

The triangle ABC (labelled anti-clockwise) has a right-angle at A and side-lengths $a\,(= BC)$, $b\,(= CA)$ and $c\,(= AB)$ where $b < c < a$. Initially, three insects are at rest, one at each vertex of ABC. At the same instant, they start to chase each other in an anti-clockwise direction around the sides of the triangle, each moving the same relative distance $\alpha\,(< 1)$ along their respective pursuit sides before pausing to review their situations. Thus the insect at C stops at point P on CA where $CP = \alpha b$; points Q on AB and R on BC are similarly defined.

(a) If triangle PQR is right-angled at Q show that it is, in fact, similar to triangle ABC.

The insects then start moving again, this time in a clockwise direction along the sides of the right-angled triangle PQR, each moving the same relative distance α along their respective pursuit sides before pausing. The chase continues forever in this manner, alternating between clockwise and anti-clockwise directions of pursuit.